Quasi-periodic Schrödinger operators :
spectral and dynamical aspects

Raphaël Krikorian

The aim of these lectures is to present the spectral theory of quasi-periodic Schrödinger operators on $\mathbb{Z}$ and its links with the dynamics of quasi-periodic cocycles.

Quasi-periodic Schrödinger operators are bounded self-adjoint operators of the form

$$H_{V,\alpha,x} \;:\; l^2(\mathbb{Z}) \ni (u_n)_{n\in\mathbb{Z}} \mapsto (u_{n+1} + u_{n-1} + V(x+n\alpha)u_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$$

where $V : \mathbb{T} = \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is a C^k function ($k \in \mathbb{N} \cup \{\infty, \omega\}$), α (the frequency) is in $\mathbb{T}\backslash\mathbb{Q}$ and x (the phase) is in $\mathbb{T}$.

Because α is irrational, the spectrum $\Sigma_{V,\alpha,x}$ of $H_{V,\alpha,x}$ is independent of the phase x. On the other hand the spectral measure $\mu_{V,\alpha,x}$ does depend on x. Our aim is to understand both the spectrum $\Sigma_{V,\alpha,x}$ as a set (Is it a Cantor set ? What is its Lebesgue measure ?) and the spectral type of $H_{V,\alpha,x}$ i.e. the decomposition of the spectral measure $\mu_{V,\alpha,x}$ as a sum $\mu_{ac}+\mu_{pp}+\mu_{sc}$ of an absolutely continuous measure (w.r.t. Lebesgue), a pure point measure and a singular continuous measure.

A good starting point is to understand the *generalized eigenvalue equation* $H_{V,\alpha,x}u = Eu$. At this point a connection is made with the theory of quasi-periodic cocycles *via* the equivalence :

$$H_{V,\alpha,x}u = Eu \quad\Longleftrightarrow\quad \forall\, n \in \mathbb{Z},\ \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = \begin{pmatrix} E - V(x+n\alpha) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}.$$

Indeed, one then has

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = \prod_{k=n}^{0} \begin{pmatrix} E - V(x+k\alpha) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix}$$

and understanding the behavior of (u_n) is related to understanding the preceding product of matrices.

This product has a dynamical meaning : if we denote $S_{E-V} = \begin{pmatrix} E - S & -1 \\ 1 & 0 \end{pmatrix}$ (the Schrödinger matrix) one introduces the so-called Schrödinger cocycle which is the dynamics

$$(a, S_{E-V}(\cdot)) : \mathbb{T} \times \mathbb{R}^2 \ni (x, y) \mapsto (x+\alpha, S_{E-V}(x)y) \to \mathbb{T} \times \mathbb{R}^2;$$

the preceding product of matrices is then $S_{E-V}^{(n)}$ where

$$(n\alpha, S_{E-V}^{(n)}) = (\alpha, S_{E-V})^n \quad \text{(n-th iterate)}.$$

These cocycles can have the following behaviors [1] :

— (UH) Uniformly hyperbolic ;

— (NUH) Non-uniformly hyperbolic ;

— (Red) Reducible, almost reducible.

I shall define these *dynamical* notions and explain what are their *spectral* counterparts.

A tentative plan is :

1. Definitions of the basic spectral concepts we are interested in. Presentation of quasi-periodic cocycles and their attached dynamical invariants (rotation number, Lyapunov exponents...). Links between these dynamical invariants and the spectral objects (Integrated density of states, Thouless formula...). Link between Spectrum and Uniform hyperbolicity.
2. Reducibility of quasi-periodic cocycles. KAM theory. Dinaburg-Sinai and Eliasson's theorems.
3. Non perturbative reducibility results and renormalization.
4. Positive Lyapunov exponents and Anderson localization.
5. Presentation of Artur Avila's global theory.

1. In the real analytic case, this can be made precise by using Avila's theory.