Lec. 5 **Constructions in the Hamiltonian context.**

(I) Context

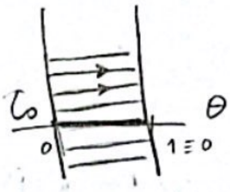
$$H_T(\theta, r) = \underbrace{\langle r, \omega_0 \rangle}_{= r_1(\omega_0)_1 + r_2(\omega_0)_2 + \dots \, = H_0(r)} + \underbrace{h(\theta, r)}_{= O(|r|^2)} \in C^\infty; \quad \theta \in \mathbb{T}^d, \; r \approx 0 \in \mathbb{R}^d$$

(H_T) — degree of freedom.

$$\begin{cases} \dot\theta_j = \partial_{r_j} H \\ \dot r_j = -\partial_{\theta_j} H \end{cases} \qquad \varphi_H^t(r, \theta) \text{ corresp. flow.}$$

$\underline{\mathcal{E}x}$ $(d=1)$ $H(r) = r\omega_0$ $\begin{cases} \dot\theta = \omega_0 \\ \dot r = 0 \end{cases}$ $\begin{cases} \theta(t) = \theta_0 + t\omega_0 \\ r(t) = r_0 \end{cases}$

⌜integrable syst. $d=1$ ⌟



$\varphi^t_{(\theta,r)} = (\theta + t\omega, r)$ ⌜$r = \text{const}$.⌟

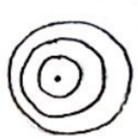
$\underline{\mathcal{E}x}$ (H_T) has invar. torus $\mathcal{T}_0 = \mathbb{T}^d \times \{0\}$; $\varphi^t_{(\theta,0)} = (\theta + t\omega_0, 0)$.

$\underline{Def}$ An invar. torus $\mathcal{T}$ is a $\underline{\text{KAM-torus}}$ if $\Phi^t|_{\mathcal{T}}$ is conjugated to $(\theta + t\omega, r)$ with $\omega \in DC$. ⌜$|\langle k, \omega \rangle| \geq \frac{\gamma}{|k|^\tau}$ for some $\gamma > 0, \tau > 1$ $\forall k \in \mathbb{Z}^d$⌟

$\underline{Rem.}$ Analogous definitions for

$$(H_p) \qquad H_p(x, y) = \underbrace{\langle r, \omega_0 \rangle}_{H_0(x,y)} + O(\|(x, y)\|^3)$$

Notation: $r = \frac{x^2 + y^2}{2}$



$\{r = 0\}$ is an elliptic fixed point (see [FS 17])

$\underline{\text{Classical KAM}}$: $\omega_0 \in DC$, "twist", $\|h\|_\sigma$ suff. small (can use C^k, C^∞ or C^ω)
$\Rightarrow \mathcal{T}_0$ is accumulated by a pos. measure set of inv. tori having Lebesgue density 1 at $\mathcal{T}_0$) $\underline{\text{KAM-stability}}$.

Different stories: C^ω / C^∞.

$\underline{\text{Conject.}}$ (Herman, ICM 98) C^ω In any nbd of a KAM-torus $\exists$ positive measure set of KAM-tori (No twist required!)

$\underline{Th}$ [EFK 15] In any nbd of a C^ω KAM-torus $\exists$ KAM-tori (unclear if positive measure).

Th [Rüssmann] C^ω, $d = 2$ for ω Diophantine — KAM-stability

Conj. (Herman, ICM98) ⌈partly proved⌋ $\boxed{C^\infty}$ $\begin{cases} d=2 \text{ stab.} \\ d\geq 3 \text{ no stab.}\end{cases}$ ②

C^∞, $d=2$ — "KAM-stability" (no Leb. dens.) ⌈$\omega_0 \in DC$, $H\in C^\infty$, $d=2$ $\Rightarrow$ $\mathcal{T}_0$ accum. by pos. measure set of KAM-tori ⌋

Analog for maps on $\mathbb{T}\times\mathbb{R}$

Herman's last geom. th. [FKri]

$\alpha\in DC$ (Lec 1)

Th(*), $d\geq 3$ (H_T) For any ω_0 (In part, $\omega_0\in DC$), $\forall\,\varepsilon>0$
$\exists\, h\in C^\infty(\mathbb{T}^d\times\mathbb{R}^d)$, $\|h\|_\infty<\varepsilon$, s.t.
the set of inv. tori has measure 0;
for a.e. r, $\forall\,\theta$ $\limsup\limits_{t\to\pm\infty}\|\Phi^t_H(\theta,r)\|=\infty$.

Focus for today.

$d\geq 4$ [EFK]
$d=3$ [B]

Th $d\geq 3$ (H_p) analogous statement, [FS17]

Q ¿How to destroy all inv. tori? — unknown.

II. Preliminaries [McDuff - Salamon]

Fact 1 Let $H(\theta,r)$ define a Hamilton. syst.,
$\Phi^t_H(\theta,r)$ — its flow
$\mathcal{U}(\theta,r)$ — a symplectic map.
Then $H\circ\mathcal{U}(\theta,r)$ defines a Hamilton. syst.,
$\Phi^t_{H\circ\mathcal{U}}(\theta,R) = \mathcal{U}^{-1}\circ\Phi^t_H(\mathcal{U}(\theta,R))$.

Fact 2 One can define a symplectic map $\mathcal{U}(\theta,r)\to(\Theta,R)$
a) a flow map of a Hamilton. system
b) by a generating function.
In part., if we want to define $\mathcal{U}$ that is close to Id, it is convenient to use gener. funct. $k(\Theta,r)=\langle\Theta,r\rangle+\dots$

Then $\begin{cases}\theta=\dfrac{\partial k}{\partial r}=\Theta+\dots \\ R=\dfrac{\partial k}{\partial\Theta}=r+\dots\end{cases}$, and $\mathcal{U}(\theta,r)=(\Theta,R)$ is a well-defined symplectic map

provided that $\left|\dfrac{\partial(\Theta,r)}{\partial(\theta,r)}\right|\neq 0$. See, e.g. [de la Llave]

III. Ideas for the pf of Th (*)

a) $\underline{d=2}$. Fix $\omega = (\omega_1, \omega_2)$ Liouville, $|\underbrace{q_1\omega_1 + q_2\omega_2}_{\langle q,\omega\rangle}|$ small, to be made precize later.
A large.

Let $k(\Theta, r) = \langle \Theta, r\rangle + \frac{A}{2\pi}\sin 2\pi(q_1\Theta_1 + q_2\Theta_2)$ be a gener. funct.,

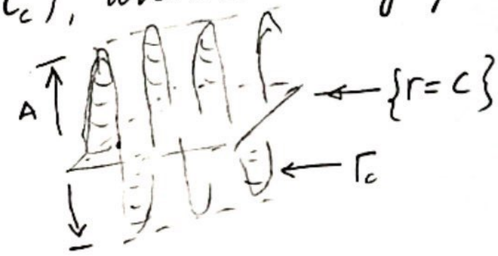
$$\begin{cases} \theta_i = \dfrac{\partial k}{\partial r_i} = \Theta_i \\[2mm] R_i = \dfrac{\partial k}{\partial \Theta_i} = r_i + Aq_i \cos 2\pi(q_1\theta_1 + q_2\theta_2); \end{cases} \qquad \text{verify } \left|\frac{\partial(\Theta, r)}{\partial(\theta, r)}\right| \neq 0$$

Then $\mathcal{U}(\theta, r) = (\Theta, R)$ is symplectic.

$$H_0 \circ \mathcal{U} = R_1\omega_1 + R_2\omega_2 = \underbrace{(r_1\omega_1 + r_2\omega_2)}_{H_0(r)} + A\underbrace{(q_1\omega_1 + q_2\omega_2)}_{\langle q,\omega\rangle \text{ small!}}\cdot \cos 2\pi(q_1\theta_1 + q_2\theta_2)$$

Given $\varepsilon > 0$, choose q so that $\| A\langle q,\omega\rangle \cos 2\pi(\langle q, \theta\rangle)\| < \varepsilon$.
Then $\| H_0 \circ \mathcal{U} - H_0 \| < \varepsilon$.

Dynamics: Let $\mathcal{T}_c = \{(\theta, r) \mid r = c = \text{const}\}$.
An invar. torus av $H_0 \circ \mathcal{U}$ is $\Gamma_c = \mathcal{U}^{-1}(\mathcal{T}_c)$, which is a graph of $\gamma(\theta) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + A\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \cdot \cos 2\pi(\langle q, \theta\rangle)$



Trajectories of $H_0 \circ \mathcal{U}$ are dense on Γ_c.

We proved:

Propos. 1 $\forall \varepsilon > 0$, $s \in \mathbb{N}$, $\Delta > 0$, $A > 0$ (and sympl. map V)
$\exists\, T > 0$ and sympl. map $\mathcal{U}$

1) $\| H_0 \circ \mathcal{U} \circ V - H_0 \circ V \|_s < \varepsilon$

2) $\sup\limits_{0<t<T} \left| \Phi^t_{H_0 \circ \mathcal{U} \circ V}(p) \right| > A$ for any p, $|p| \le \Delta$

Iterate this construction:

Let $H_{n-1} = H_0 \circ \underbrace{\mathcal{U}_{n-1} \circ \dots \mathcal{U}_1}_{V_{n-1}}$, $\varepsilon_n = \frac{1}{2^n}$, $A_n = 2^n$, $S_n = 2^n$

Construct $\mathcal{U}_n$ (and $(q_1^{(n)}, q_2^{(n)})$) so that

$$H_n = H_0 \circ \mathcal{U}_n \circ V_{n-1} \underset{\text{in } C^{S_n}}{\overset{\varepsilon_n}{\approx}} H_0 \circ V_{n-1} = H_{n-1}$$

- $(H_n)_n$ form a Cauchy sequence in any $C^{S_n} \Rightarrow$ converge to $H \in C^\infty$; and $\underbrace{H \overset{\varepsilon_n}{\approx} H_n}$.
- Invar. tori of H_n have the form $(\mathcal{U}_n \circ V_{n-1})^{-1}(\mathcal{T}_c) := \Gamma_{n,c}$ (very wiggly).
- $\forall \Delta > 0$, $|p| < \Delta$, we have: $\varphi_H^t(p)$ is "very close" to $\varphi_{H_n}^t(p)$ for $\forall A > 0$ a "very long" time. Hence, $\exists\, T = T(\Delta, A)$ s.t. $\sup\limits_{0<t<T} \min\limits_{|p|<\Delta} |\varphi_H^t(p)| > A$.

One can also prove the converg. "a la Herman" (Baire cathegory th.)

Let $\mathcal{H}_0 = \{ H_0 \circ \mathcal{U} \mid \mathcal{U} \text{ symplectic} \}$, $\overline{\mathcal{H}_0}$ ← in C^∞

Propos. The set $D = \{ H \in \overline{\mathcal{H}_0} \mid \limsup\limits_{t \to \infty} |\varphi_H^t(p)| = \infty \ \ \forall p \in (0, r) \}$ is a dense G_δ-subset of $\overline{\mathcal{H}_0}$

Pf For any $\Delta > 0$, $A > 0$, $T \in \mathbb{N}$, let

$$\mathcal{D}(\Delta, A, T) = \{ H \in \overline{\mathcal{H}_0} \mid \sup_{0<t<T} \min_{|p|<\Delta} |\varphi_H^t(p)| > A \}$$

↑ open.

By Prop. 1, $\forall A, \forall \Delta$, $\bigcup\limits_{T \in \mathbb{N}} D(\Delta, A, T)$ is dense in $\overline{\mathcal{H}_0}$.

Take $A_n = 2^n$, $\Delta_n = 2^{-n}$

$$D = \bigcap_{A_n} \bigcap_{\Delta_n} \underbrace{\left(\bigcup_{T \in \mathbb{N}} D(\Delta_n, A_n, T) \right)}_{\text{open dense}}$$ is a G_δ-set (dense).

We showed: for $d=2$, $\omega_0 \,(\in \mathbb{R}^2)$ Liouville, $H_0 = \langle r, \omega_0 \rangle$

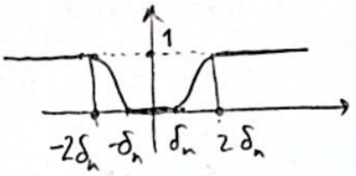
the map $H = \lim_{n\to\infty} H_0 \circ \mathcal{U}_n \circ \mathcal{U}_{n-1} \ldots \mathcal{U}_1 = \langle r, \omega_0 \rangle + \underline{\underline{\sum_n^{\infty} A_n \cos\left(2\pi\left(q_1^{(n)}\theta_1 + q_2^{(n)}\theta_2\right)\right)}}$

exists, $H \in C^\infty$, and has no invariant tori.

How to keep an invar. torus?

<u>Idea 1</u> (parameter). Take $d=3$, $\omega_0 = (\omega_1, \omega_2, 1)$,

$\underset{\mathbb{R}^3 \ni}{H_0 = \langle r}, \omega_0 \rangle$, let $\underset{C^\infty}{\underset{\Uparrow}{a_n(r_3)}} =$



$\delta_n = \frac{1}{2^n}$.

Let $H(\theta, r) = \langle r, \omega_0 \rangle + \underline{\underline{\sum A_n \cdot \underline{a_n(r_3)} \cos 2\pi\left(q_1^{(n)}\theta_1 + q_2^{(n)}\theta_2\right)}}$.

(compare with the above!). ↖ no θ_3 here!

<u>Note</u> $\dot{r}_3 = \frac{\partial H}{\partial \theta_3} = 0 \Rightarrow r_3(t) = r_3(0)$ - <u>parameter</u>

For each fixed r_3 investigate the corresp. system.

If $r_3 = c \neq 0$, then $H\big|_{\{r_3 = c\}}$ has no invariant tori (as before)

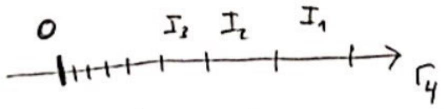
For $r_3 = 0$: $H\big|_{r_3=0} = \omega_1 r_1 + \omega_2 r_2$; In particular, $T_0 = \{r = 0\}$ is an invariant torus.

<u>Idea 2</u> Prove the same result for <u>any</u> $\omega_0 \in \mathbb{R}^4$.

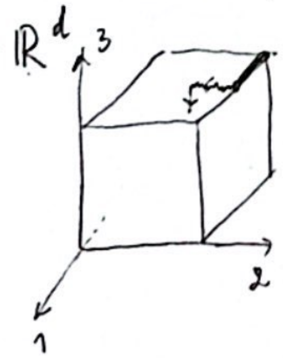
Fix ω_0 (normalize: $\omega_0 = (\omega_1, \omega_2, \omega_3, 1)$), let $H_0 = \langle r, \omega_0 \rangle$ (r_4 - parameter).

Change $H_0 \rightsquigarrow \tilde{H}_0 = (\omega_1 + f_1(r_4), \omega_2 + f_2(r_4), \omega_3 + f_3(r_4), 1)$

where $f_i(r_4) \xrightarrow[r_4 = 0]{} 0$ so that:



$\omega \in \mathbb{R}^{d}$, 3



- $\forall r_4 \in I_{1+k\cdot 3}$, $(\omega_2 + f_2(r_4), \omega_3 + f_3(r_4))$ is a constant Liouville vector, $\omega_1 + f_1(r_4)$ changes
- $\forall r_4 \in I_{2+k\cdot 3}$, $(\omega_1 + f_1(r_4), \omega_3 + f_3(r_4))$ is a constant Liouville vector, $\omega_2 + f_2(r_4)$ changes;

• for $r_4 \in I_{3k}$, $(\omega_1 + f_1(r_4), \omega_2 + f_2(r_4))$ is a constant Liouville vector, $\omega_3 + f_3(r_4)$ changes,

and $\boxed{\lim_{r_4 \to 0} (\omega_1 + f_1(r_4), \omega_2 + f_2(r_4), \omega_3 + f_3(r_4), 1) \to \omega_0.}$

Make all the above smooth (see [EFK 15])

For each r_4 construct H using the Liouville pair of frequences (which one – depends on r_4) and independent on the changing frequency.
This can be done smoothly.