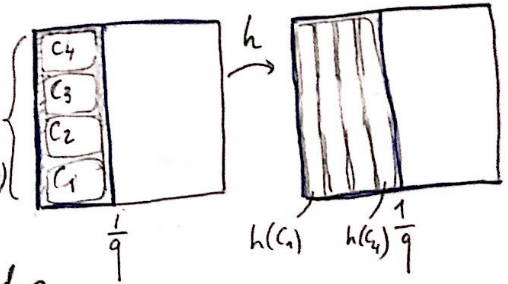
Lec 4 Examples of AbC constructions II.

<u>Ex</u> $\forall \frac{p}{q}, \forall \ell \in \mathbb{N}, \forall \delta > 0$
$\exists\, h \in \mathrm{Diff}^\infty(\mathbb{A}, \mu)$



$\frac{1}{q}$-periodic in x,
$h|_{Blue} = id$

(Yesterday we constructed a specific h that satisfied, moreover,
- h = isometry on a set of measure $1-\delta$
- for any k, $\|h\|_k \le C(k,\delta) \cdot q^k$)

<u>Note</u> that the paring of $\longrightarrow \boxed{j} \to$ can be chosen different.

$f_1 := h^{-1} \circ R_{\frac{p}{q} + \frac{r}{q\ell}} h$ permutes elements $\boxed{C_1}, \ldots \boxed{C_{q\ell}}$ cyclically.
$\frac{p_2}{q_2}$

For any $H \in \mathrm{Diff}^\infty(\mathbb{A}, \mu)$, q and ℓ can be chosen so large (depending on H) that
$f_1 := H \circ h^{-1} \circ R_{\frac{p_2}{q_2}} \circ h \circ H$ permutes elements
$H^{-1}(\boxed{C_1}) \ldots H^{-1}\boxed{C_{q\ell}}$ cyclically.

For a sequence $\frac{p_n}{q_n} \to \alpha$, q_n "are chosen sufficiently large" at each step,

construct $f_n = \underbrace{h_1^{-1} \ldots h_n^{-1}}\, R_{\alpha_{n+1}} \underbrace{h_n \ldots h_1}_{H_n}$.

$\forall \varepsilon > 0$, if $\frac{p_n}{q_n} \to \alpha$ "suff. fast", then $\lim_{n\to\infty} f_n = f \in \mathrm{Diff}^\infty(\mathbb{A}, \mu)$, $\underline{\|f - R_\alpha\|_\infty < \varepsilon}$.

(<u>We saw</u>: If $\|H_n\| \le C(n) \cdot q_n^{p(n)}$, where $p(n)$ is a polynomial, then $\forall$ Liouville α $\exists$ a sequence $\frac{p_n}{q_n} \to \alpha$ s.t. the limit f exists.)

(II) Speed of approximation

Def [Katok-Stepin], [Katok-Robinson]

Let $S(n)$ be a monotonic sequence, $S(n) \xrightarrow[n\to\infty]{} 0$.

Automorphism f of (M,μ) admits a cyclic approximation by periodic transformations (c.a.p.t.) with speed $S(n)$ if $\exists$ a sequence of partitions $\xi_n = (C_{i,n})_{i=1}^{q_n}$ and autom:s f_n such that f_n permutes $C_{i,n}$ cyclically and

- $\xi_n \to \varepsilon$
- $\sum_{i=1}^{q_n} \mu(f(C_{n,i})) \Delta f_n(C_{n,i}) < S(q_n)$.

Th ([Katok-Stepin]) If an automorphism f admits c.a.p.t with speed $\frac{\theta}{n}$ for $\theta < 4$, then f is ergodic.

Th ([AK70]) For any $S(n) \xrightarrow[n\to\infty]{} 0$, the set of autom:s in $\mathcal{A}_\alpha^\infty$ admitting c.a.p.t. with speed $S(n)$ is generic.

Recall $f \in \mathcal{D}iff(M,\mu)$ is mixing if $\forall A, B \subset M$ (measurable)

$$\lim_{n\to\infty} \mu(f^n(A) \cap B) = \mu(A)\mu(B)$$

"close" to capt

Th ([Katok-Stepin]) If an autom. f admits "capt II" with speed $\frac{\theta}{n}$, $\theta < 2$, then f is not mixing.

Corollary: Ergodic but not mixing diffeos are generic in $\mathcal{A}_\alpha^\infty$

Why generic f is not mixing: Rigidity, i.e.

$f^{q_n} \overset{\varepsilon_n}{\approx} f_{n-1}^{q_n} = id$, so $\exists (q_n) \to \infty$ s.t. $f^{q_n} \xrightarrow[\text{(uniform)}]{} id$.

Q Does $\exists$ a mixing $f \in \mathcal{D}iff^\infty(\mathbb{D}, \mu)$?

Known: $\nexists$ mixing $f \in \mathcal{D}iff^\omega(\mathbb{D}, \mu)$ [AFLXZ]

$\exists$ mixing $f \in \mathcal{D}iff^\omega(\mathbb{T}^{\geq 3}, \mu)$ [Fayad 2000]

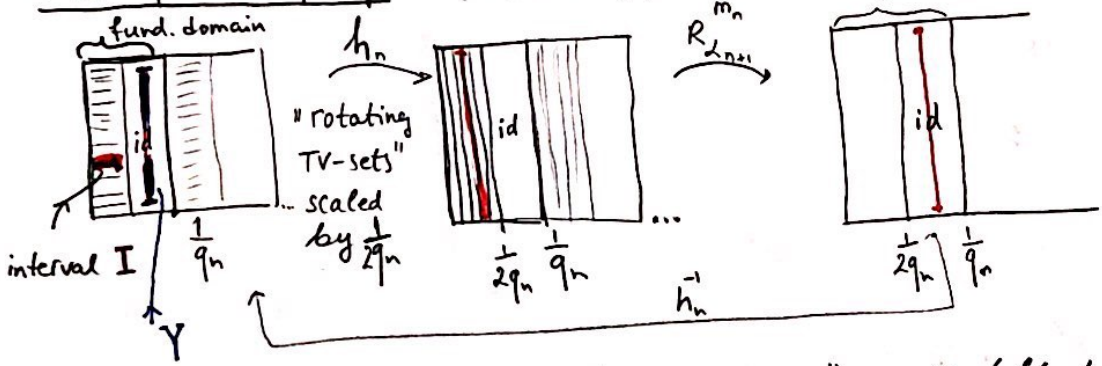
Def $f \in \mathrm{Diff}(M,\mu)$ is <u>weakly mixing</u> if $\exists$ sequence $m_n \xrightarrow[n\to\infty]{} \infty$ s.t. $\forall$ measur. $A, B \subset M$ we have
$$\mu(f^{m_n}(A) \cap B) \xrightarrow[n\to\infty]{} \mu(A)\mu(B)$$
(see [Sklover] for this variant of the definition)

Th [AK70], [FS05] $\forall$ Liouville α

weakly mixing diffeos are generic in $\mathcal{A}^\infty_\alpha(M,\mu)$ ← $\mathbb{T}^{\geq 2}$, $\mathbb{A}$, $\mathbb{D}$

Rem. Rigidity + weakly mixing. [Kunde 15]

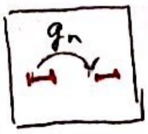
Idea of the proof (one step)



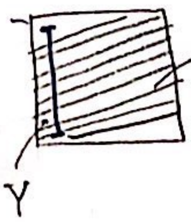
- Define h_n as "scaled rotated TV-set" on the left half of the fundam. domain, and $h_n = \mathrm{id}$ on the right half of the fund. domain, h_n is $\frac{1}{q_n}$-periodic in x.
- Choose $m_n \leq q_{n+1}$ s.t. $m_n \alpha_{n+1} \pmod 1$ is $\frac{1}{q_{n+1}}$-close to $\frac{1}{2q_n}$.
- Define $\varphi_n = h_n^{-1} R_{\alpha_{n+1}} h_n$.

Then $\varphi_n^{m_n}(\text{—}) = \varphi^{m_n}(I) = Y = \text{|}$ see figure.
↑ interval, see figure

- Define $g_n(x,y) = (x + Ay, y)$ $(A = nq_n)$

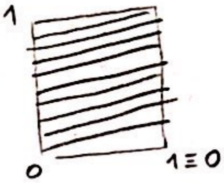


then $g_n^{-1}(x,y) = (x - Ay, y)$



$g_n^{-1}(Y)$ is very well-distributed on $\mathbb{A}$

- Let $\tilde{f}_n = g_n^{-1} h_n^{-1} R_{\alpha_{n+1}} h_n g_n$

then $\tilde{f}_n^{m_n}(\text{—}) =$  The image is $\leq \frac{1}{q_n}$-dense on $\mathbb{A}$

- Finally, let $f_n = \underbrace{H_{n-1}^{-1}}_{\substack{\text{size defined} \\ \text{by } q_1, \dots q_{n-1}}} \tilde{f}_n \underbrace{H_{n-1}}_{-\shortparallel-}$

If $q_n \gg q_{n-1}$, $f_n^{m_n}(\text{—})$ is $\frac{1}{2^n}$-dense on $\mathbb{A}$.

This implies (approximating boxes A by intervals, using Fubini th.)
$\mu(f_n^{m_n}(A) \cap B) \approx \mu(A)\mu(B)$.

<u>Note</u>: $\|h_n\|_k \leq q_n^k$ (since h_n is a standard transf. scaled by q_n)

$\|g_n\|_k \leq q_n^k$ (explicit)

As remarked before, this is enough to run the construction in $\mathcal{A}_\alpha^\infty$ for any $\alpha \in$ Liouvill.

III <u>Other examples</u>

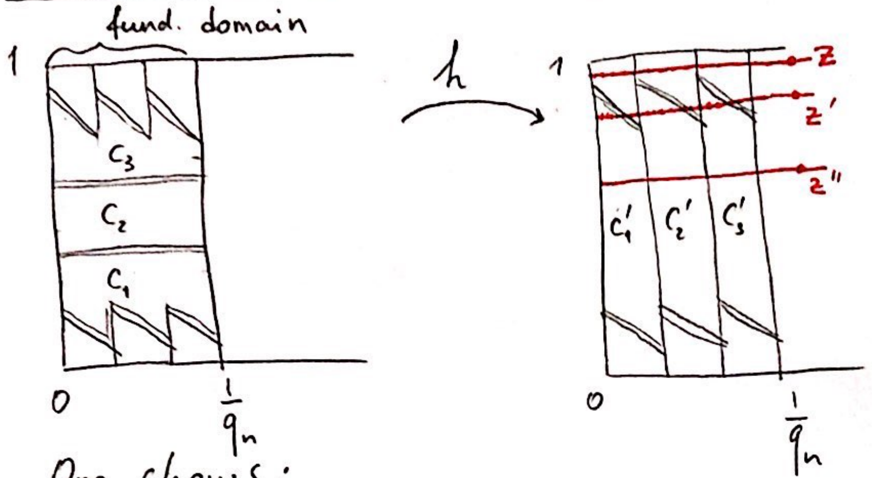
Any area-pres. homeo of $\mathbb{D}$ has at least 3 ergodic invar. measures (at the fixed pt, on $\partial\mathbb{D}$ and on $\mathring{\mathbb{D}}$).

<u>Th</u> [FK] $\exists f \in \mathrm{Diff}^\infty(\mathbb{D}, \mu)$ that has exactly 3 ergodic invariant measures.

<u>Rem</u>: One can construct f with more measures.

<u>Th.</u> [Windsor] $\exists f \in \mathrm{Diff}^\infty(\mathbb{T}^2, \mu)$ which is <u>minimal</u> and has 2 inv. measures (both a.c. w.r.t. Lebesgue)

Idea [FK] (one step, on $\mathbb{A}$)



$f_n := h_n^{-1} R_{\alpha_{n+1}} h_n$

$\left.\begin{array}{l}\delta_u \\ \delta_l\end{array}\right\}$ 1-dim. measures on the boundaries

$\mu = \text{Leb.}$

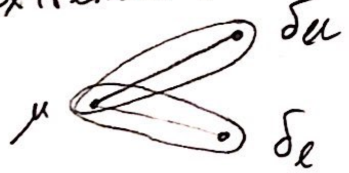
$\mu(C_j) = \mu(C_i) \quad \forall i,j$

One shows:

$\forall z \in \mathbb{A}, \quad \forall \varphi \in C^\infty(\mathbb{A}) \quad \exists \quad (m_n) \to \infty, \ (\varepsilon_n) \to 0$ s.t.

$$\frac{\sum_{k=1}^{m_n-1} \varphi(f_n^k(z))}{m_n} \overset{\varepsilon_n}{\approx} a_n(z) \int \varphi \, d\mu + (1 - a_n(z)) \int \varphi \, d\delta_u \text{ (or } \delta_l\text{)}$$

We see that any invar measure is a lin. combin. of μ and δ_u (or μ and δ_l) with μ, δ_u, δ_l as extrema:



Ergodic measures are the extremal ones, i.e μ, δ_u and δ_l.