Lec. 3 Examples of AbC constructions.

(I) We say that $f_1 \in \mathrm{Diff}^\infty(M_1, \mu_1)$ is a smooth realization of dynamics f_2 on (M_2, μ_2) if f_1 and f_2 are measure-theoretically isomorphic, i.e. $\exists$ an a.s. one-to-one map $K: M_1 \to M_2$ s.t. $K \circ f_1 = f_2 \circ K$, and $\mu_1(K^{-1}(A)) = \mu_2(A)$ $\forall \mu_2$-meas. set A.

⌈This was a motivation for [AK70]⌋

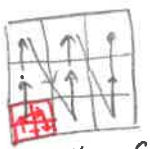
Th. [AK70] $\forall \varepsilon > 0$ $\exists \alpha$ and $\exists f \in \mathrm{Diff}^\infty(M, \mu)$ such that

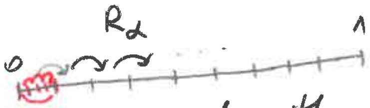
(For any Liouville α by [FSW])

cpt, supporting a periodic flow

f is measure-theoret. isomorphic to $R_\alpha : \mathbb{T}^1 \circlearrowleft$

and $\| f - R_\alpha \|_\infty < \varepsilon$.

f  μ_1-area

R_α  μ_2 - length

Def A sequence of partitions ξ_n monotone if ξ_{n+1} is a refinement of ξ_n and generating ($(\xi_n) \to \varepsilon$) if $\{x\} = \bigcap_{n=1}^{\infty} C_n(x)$ $\forall x \in M'$ s.t. $\mu(M \setminus M') = 0$

element of ξ_n contain. x.

Lemma [AK70] (and many other places).

Let: M_1, M_2 - Lebesgue spaces,

$(\xi_n^{(1)}), (\xi_n^{(2)})$ generating seq.'s of partitions of M_1, M_2, resp.

$(f_n^{(1)}), (f_n^{(2)})$ sequences of autom. of M_1, M_2, resp. s.t.

$f_n^{(i)} \xrightarrow[n\to\infty]{\text{weak}} f^{(i)}$ $(i = 1, 2)$,

Supp. $\forall n$ $\exists$ measure-theor. isomorph. K_n such that

$K_n : M_1/\xi_n^{(1)} \to M_2/\xi_n^{(2)}$

- $f_n^{(2)}|_{\xi_n^{(2)}} \circ K_n = K_n \circ f_n^{(1)}|_{\xi_n^{(1)}}$
- $\forall \Delta \in \xi_{n-1}^{(1)}$ $K_n(\Delta) = K_{n-1}(\Delta)$

Then $f^{(1)}$ and $f^{(2)}$ are meas-theor. isomorphic.

II Speed of approximation

Def [Katok-Stepin], [Katok-Robinson]

Let $S(n)$ be a monotonic sequence, $S(n) \xrightarrow[n\to\infty]{} 0$

Automorphism f of (M,μ) admits a cyclic approximation by periodic transformations (c.a.p.t.) with speed $S(n)$ if $\exists$ a sequence of partitions $\xi_n = (C_{i,n})_{i=1}^{q_n}$ and autom:s f_n such that f_n permutes $C_{i,n}$ cyclically and

- $\xi_n \to \varepsilon$
- $\sum_{i=1}^{q_n} \mu(f(C_{n,i})) \Delta f_n(C_{n,i}) < S(q_n)$.

Th ([Katok-Stepin]) If an automorphism f admits c.a.p.t with speed $\frac{\theta}{n}$ for $\theta < 4$, then f is ergodic.

Th ([AK70]) For any $S(n) \xrightarrow[n\to\infty]{} 0$, the set of autom:s in $\mathcal{A}_\alpha^\infty$ admitting c.a.p.t. with speed $S(n)$ is generic.

Recall $f \in \mathcal{D}iff(M,\mu)$ is mixing if $\forall A, B \subset M$ (measurable)

$$\lim_{n\to\infty} \mu(f^n(A) \cap B) = \mu(A)\mu(B)$$

"close" to capt

Th ([Katok-Stepin]) If an autom. f admits "capt II" with speed $\frac{\theta}{n}$, $\theta < 2$, then f is not mixing.

Corollary: Ergodic but not mixing diffeos are generic in $\mathcal{A}_\alpha^\infty$

Why generic f is not mixing: Rigidity, i.e.

$$\boxed{f}^{q_n} \overset{\varepsilon_n}{\approx} f_{n-1}^{q_n} = \underline{id}, \text{ so } \exists (q_n) \to \infty \text{ s.t. } f^{q_n} \xrightarrow[\text{(uniform)}]{} id.$$

Q Does $\exists$ a mixing $f \in \mathcal{D}iff^\infty(\mathbb{D}, \mu)$?

Known: $\nexists$ mixing $f \in \mathcal{D}iff^\omega(\mathbb{D}, \mu)$ [AFLXZ]

$\exists$ mixing $f \in \mathcal{D}iff^\omega(\mathbb{T}^{\geq 3}, \mu)$ [Fayad 2000]

Def $f \in \mathrm{Diff}(M,\mu)$ is weakly mixing if $\exists$ sequence $m_n \xrightarrow[n\to\infty]{} \infty$ s.t. $\forall$ measur. $A, B \subset M$ we have

$$\mu\left(f^{m_n}(A) \cap B\right) \xrightarrow[n\to\infty]{} \mu(A)\mu(B)$$

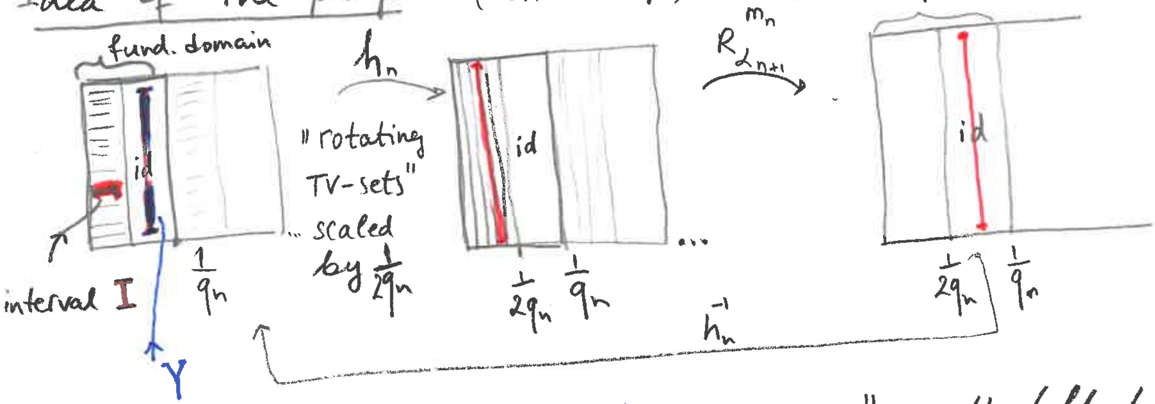
(see [Sklover] for this variant of the definition)

Th [AK70], [FS05] $\forall$ Liouville α

weakly mixing diffeos are generic in $\mathcal{A}_\alpha^\infty(M,\mu)$ ← $\mathbb{T}^{\geq 2}$, $\mathbb{A}$, $\mathbb{D}$

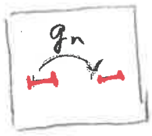
Rem. Rigidity + weakly mixing. [Kunde 15]

Idea of the proof (one step)



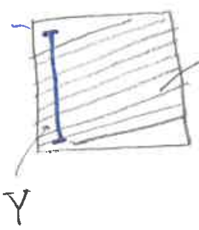
- Define h_n as "scaled rotated TV-set" on the left half of the fundam. domain, and $h_n = \mathrm{id}$ on the right half of the fund. domain, h_n is $\frac{1}{q_n}$-periodic in x.
- Choose $m_n \leq q_{n+1}$ s.t. $m_n \alpha_{n+1} \pmod 1$ is $\frac{1}{q_{n+1}}$-close to $\frac{1}{2q_n}$.
- Define $\Phi_n = h_n^{-1} R_{\alpha_{n+1}} h_n$.

  Then $\Phi_n^{m_n}(\;) = \Phi_n^{m_n}(I) = Y =$ see figure.
  ↑ interval, see figure



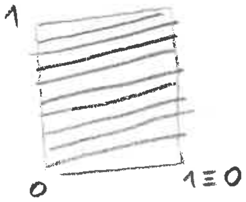
- Define $g_n(x,y) = (x + Ay, y)$
  $(A = nq_n)$

  then $g_n^{-1}(x,y) = (x - Ay, y)$



$g_n^{-1}(Y)$ is very well-distributed on A

- Let $\tilde{f}_n = g_n^{-1} h_n^{-1} R_{\alpha_{n+1}} h_n g_n$

then $\tilde{f}_n^{m_n}(-) =$ 

The image is $\leq \frac{1}{q_n}$-dense on $\mathbb{A}$

- Finally, let $f_n = \underbrace{H_{n-1}^{-1}}_{\text{size defined by } q_1, \dots q_{n-1}} \tilde{f}_n \underbrace{H_{n-1}}_{-''-}$

If $q_n \gg q_{n-1}$, $f_n^{m_n}(\text{—})$ is $\frac{1}{2^n}$-dense on $\mathbb{A}$.

This implies (approximating boxes A by intervals, using Fubini th)
$\mu(f_n^{m_n}(A) \cap B) \approx \mu(A)\mu(B)$.

<u>Note</u>: $\|h_n\|_k \leq q_n^k$ (since h_n is a standard transf. scaled by q_n)

$\|g_n\|_k \leq q_n^k$ (explicit)

As remarked before, this is enough to run the construction in $\overline{\mathcal{A}_\alpha^\infty}$ for any $\alpha \in$ Liouvill.

III <u>Other examples</u>

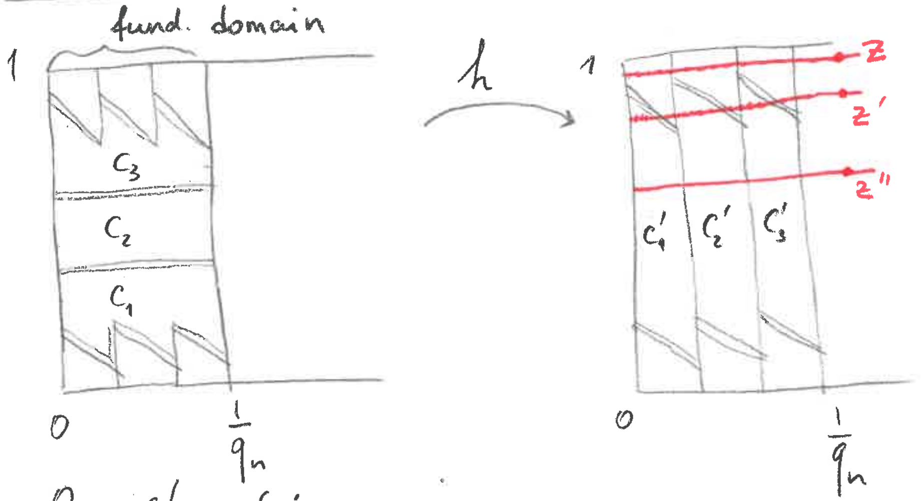
Any area-pres. homeo of $\mathbb{D}$ has at least 3 ergodic invar. measures (at the fixed pt, on $\partial\mathbb{D}$ and on $\mathring{\mathbb{D}}$).

<u>Th</u> [FK] $\exists f \in \mathrm{Diff}^\infty(\mathbb{D}, \mu)$ that has exactly 3 ergodic invariant measures.

<u>Rem</u>: One can construct f with more measures.

<u>Th.</u> [Windsor] $\exists f \in \mathrm{Diff}^\infty(\mathbb{T}^2, \mu)$ which is <u>minimal</u> and has 2 inv. measures (both a.c. w.r.t. Lebesgue)

Idea [FK] (one step, on $\mathbb{A}$)



$f_n := h_n^{-1} R_{\alpha_{n+1}} h_n$

$\left.\begin{matrix}\delta_u \\ \delta_l\end{matrix}\right\}$ 1-dim. measures on the boundaries

$\mu = \text{Leb.}$

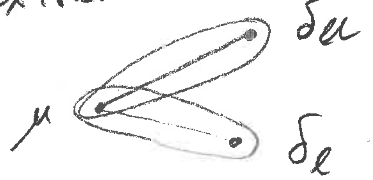
$\mu(C_j) = \mu(C_i) \quad \forall i,j$

One shows:

$\forall z \in \mathbb{A}$, $\forall \varphi \in C^\infty(\mathbb{A})$ $\exists$ $(m_n) \to \infty$, $(\varepsilon_n) \to 0$ s.t.

$$\frac{\sum_{k=1}^{m_n-1} \varphi(f_n^k(z))}{m_n} \overset{\varepsilon_n}{\approx} a_n(z) \int \varphi \, d\mu + (1 - a_n(z)) \int \varphi \, d\delta_u \text{ or } \delta_l.$$

We see that any invar. measure is a lin. combin. of μ and δ_u (or μ and δ_l) with μ, δ_u, δ_l as extrema:



Ergodic measures are the extremal ones, i.e. μ, δ_u and δ_l.