# Lec 2 Combinatorics via partitions

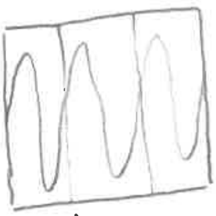
## I) Rep. + some new ideas.

a) **Lemma *** On $(\mathbb{T}^2, \mu)$, for any $\varepsilon > 0$, $\alpha_0 = \frac{p_0}{q_0}$, $k > 0$ (or $\varrho$)
($\mu$: Lebesgue)

$h \in \mathrm{Diff}^{k, \infty \text{ or } \omega}(\mathbb{T}^2, \mu)$, $m \geq 0$ $\exists$ open interval $\Delta = \Delta(\varepsilon, k, \alpha_0, h, m)$ centered at $\alpha_0$

s.t. $\forall \alpha_1 \in \Delta$ we have:

- $\| \underbrace{h^{-1} R_{\alpha_1} h}_{f_1} - \underbrace{R_{\alpha_0}}_{f_0 \text{ (or } \varrho)} \|_k < \varepsilon$
- $\| f_1^j - f_0^j \|_0 < \varepsilon \quad \forall j = 0, \dots m$



$\frac{1}{q_0}$

Iterations:

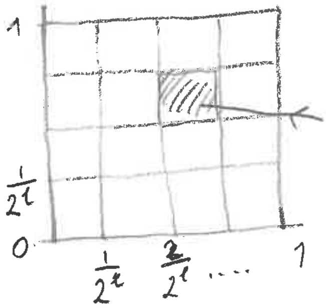
$$f_{n-1} = H_{n-1}^{-1} R_{\alpha_n} H_{n-1}$$
$$f_n = H_{n-1}^{-1} h_n^{-1} R_{\alpha_{n+1}} h_n H_{n-1}$$
$\varepsilon_n$ $\approx$

$f_n \xrightarrow{C^{k,\infty,\omega}} f$; $\|f - R_{\alpha_0}\| < \varepsilon$; $\forall n$, $\|f^j - f_n^j\|_0 < \varepsilon_n$ for $j = 1, \dots m_{n+1}$

b) Let $\mathcal{A}_\alpha^\infty = \{ h^{-1} \circ R_\alpha \circ h \mid h \in \mathrm{Diff}^\infty(M, \mu) \}$.

**Th** The set $\mathcal{M}_\alpha = \{ f \in \overline{\mathcal{A}_\alpha^\infty}(\mathbb{T}^2, \mu) \mid f \text{ is minimal on } \mathbb{T}^2 \}$ is generic in $\overline{\mathcal{A}_\alpha^\infty}$ (i.e. contains a dense $G_\delta$-set).

**Pf** (à la Herman). Fix $l$, $T$




$\mathcal{M}_\alpha(l, T) := \{ f \in \overline{\mathcal{A}_\alpha^\infty} \mid \forall x \in \mathbb{T}^2, \text{ the orbit } \{f^j(x)\}_{j=0}^T \text{ visits every } B_{ij}(l) \}$

Open. $\bigcup_{T \geq 1} \mathcal{M}_\alpha(l, T)$ is open, dense (yesterday).

$\mathcal{M}_\alpha$ contains $\bigcap_{l \geq 1} \Big( \underbrace{\bigcup_{T \geq 1} \mathcal{M}_\alpha(l, T)}_{\text{open, dense}} \Big)$, which is a $G_\delta$-set (in part., non-empty.)

c) <u>What $\alpha$ works?</u> Usually, need $\frac{p_n}{q_n} \xrightarrow{\text{fast}} \alpha$.

Case $C^\infty$: $\|f_n - f_{n-1}\|_{k_n} = \| H_{n-1}^{-1} h_n^{-1} R_{\alpha_{n+1}} h_n H_{n-1} - H_{n-1}^{-1} \underbrace{\circ}_{h_n^{-1}} \circ R_{\alpha_n} \circ_{h_n} H_{n-1} \|_k \le$

($k_n \nearrow \infty$)

$$\|\underbrace{H_{n-1}}_{\ll q_n} \circ \underbrace{h_n}_{\cos 2\pi q_n x}\|_{k+1}^{k+1} \cdot |\alpha_n - \alpha_{n+1}| \underset{\text{in our case of yesterday}}{\le} q_n^{2(k+1)^2} \cdot \underbrace{|\alpha_n - \alpha_{n+1}|}_{\le 2|\alpha - \alpha_n|}.$$

Suppose $\alpha$ is Liouville: $\forall \tau, \gamma \ \exists\, p, q$ s.t. $|\alpha - \frac{p}{q}| < \frac{\gamma}{q^\tau}$.

Then $\exists\, \alpha_n = \frac{p_n}{q_n}$ s.t. $|\alpha - \frac{p_n}{q_n}| < \frac{\varepsilon}{2^{n+1} \cdot q^{2(k+1)^2}}$,

and hence $\|f_n - f_{n-1}\|_{k_n} < q_n^{2(k+1)^2} \cdot \frac{\varepsilon}{2^{n+1} q^{2(k+1)^2}} < \frac{\varepsilon}{2^{n+1}}$,

and $f_n \to f \in C^\infty$.

We see: for any Liouville $\alpha$ we can do $\underline{\underline{C}}^\infty$ constructions in $\mathcal{A}_\alpha^\infty$ provided that $h_n$ has a polynomial estimate in terms of $q_n$. This is what we do in [FS05].

d) <u>What manifolds:</u> (see [AK70]

We need: $M$ is smooth, cpt an supports a non-trivial periodic flow: $x \mapsto S_t(x)$ s.t. $S_t \neq id$ for all $t$ and $\exists\, T > 0$ s.t. $S_T = id$.

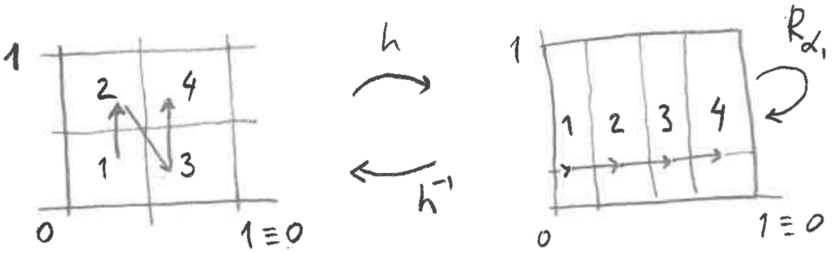
In part., $M = \mathbb{T}^d$, $\mathbb{A} = [0,1]^d \times \mathbb{T}$, $\mathbb{D}$, $S^1$,

$\underset{\text{cpt}}{M_1} \times \mathbb{T}$.

## (II) <u>Combinatorics via partitions</u>

Note: we had explicit formulas; good on $\mathbb{T}^d$.
Other manifolds?

a) Idea (discontin.) Let $M = \mathbb{A}$




$f = h^{-1} R_{\alpha_1} h \approx R_{\alpha_0}$ if $\alpha_1 \approx \alpha_0$

Let $\alpha_0 = 0$, $R_{\alpha_0} = id$

$\alpha_1 = \frac{1}{4}$ $\rightsquigarrow$ $f$ permutes elements of particiton $\xi$

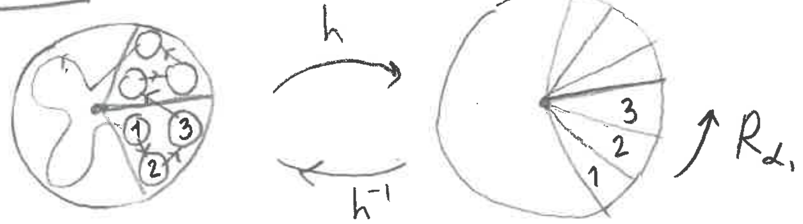
or $\alpha_1 = \frac{1}{400}$ (Note: $f^{100} = h^{-1} R_{\alpha_1}^{100} h$ permutes $\xi$)

or $\alpha_1 \notin \mathbb{Q}$.

Given $\varepsilon > 0$, $m \in \mathbb{N}$ we want to:

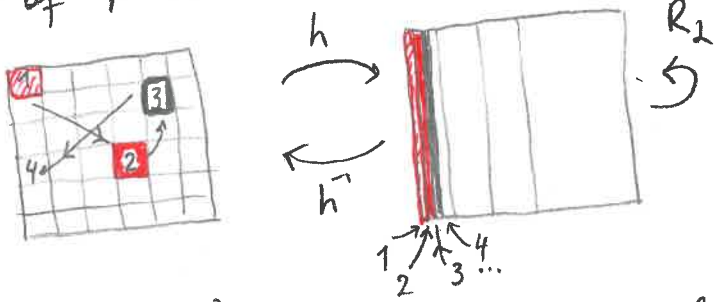
- Produce smooth area-preserving $h$ with the desired combinatorics
- Take $\alpha_1 \approx \alpha_0$ so that $\| h^{-1} R_{\alpha_1}^j h - R_{\alpha_0}^j \| < \varepsilon$ $\forall j = 0..m$.

Remark: • Classical picture: [AK70]



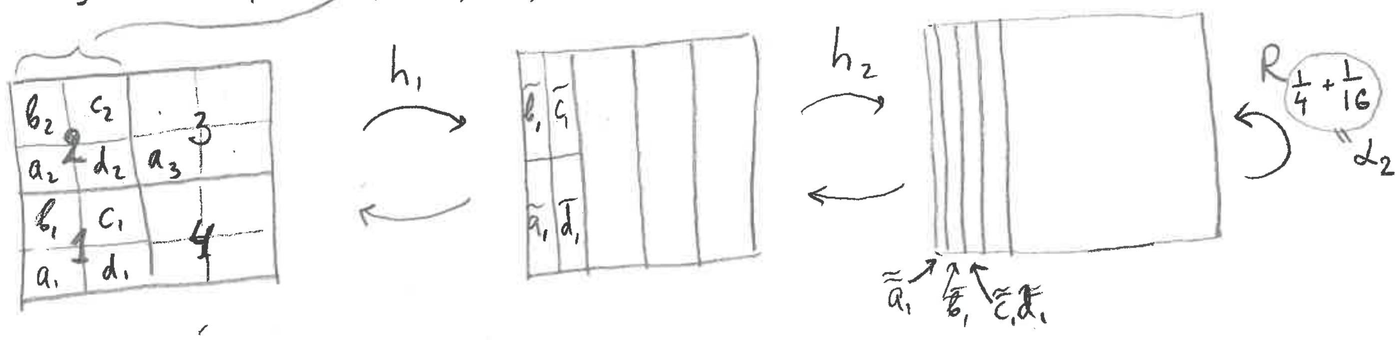

- A lot of possibilities!

E.g.




One step.

- E.g. $\xi_1 = \{ C_j \}_{j=1..4}$; $\xi_2$ is a refinement of $\xi_1$




$R_{\frac{1}{4} + \frac{1}{16}}$, $= \alpha_2$

Let $f_2 = h_1^{-1} h_2^{-1} R_{\alpha_2} h_2 h_1$. $f_2$: $a_1 \to b_2$, $b_1 \to c_2$, ..... $\frac{1}{16}$-periodic.

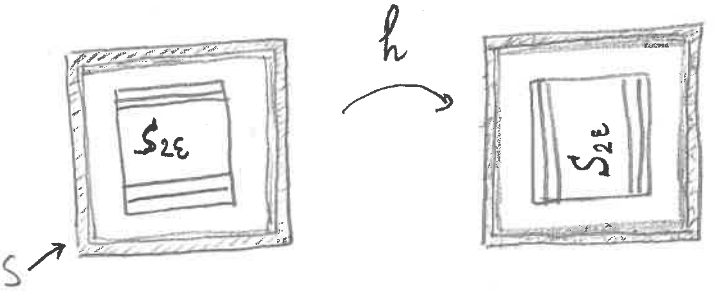
Play with this!

## b) An example: $h \in C^\infty$, area-preserving.

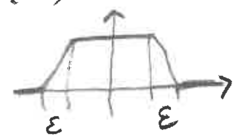


**Lemma** [FS 05]. Let $S = [0,1] \times [0,1]$, $S_\varepsilon = [\varepsilon, 1-\varepsilon]^2$

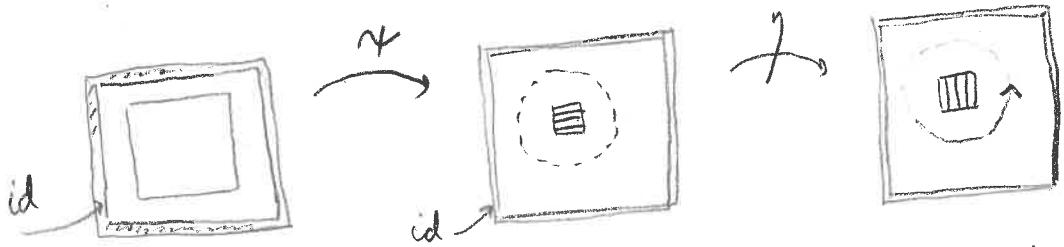
$\exists\, h \in \mathrm{Diff}^\infty(S, \mu)$ s.t. ($\mu$: area)




- $h|_{S \setminus S_\varepsilon} = id$
- $h$ rotates $S_{2\varepsilon}$ (isometry) by $\frac{\pi}{2}$.

**Pf.** Let
$$\psi(x,y) = \begin{cases} (x,y) & \text{on } S \setminus S_\varepsilon \\ \frac{x}{5}, \frac{y}{5} & \text{on } S_{2\varepsilon}. \end{cases}$$
$\psi \in \mathrm{Diff}^\infty(S)$ (think of  )


$$\eta(x,y) = \begin{cases} (y, -x) & \text{on } \sqrt{x^2+y^2} \le \frac{1}{3} \\ (x,y) & \text{on } \sqrt{x^2+y^2} \ge \frac{2}{3}. \end{cases}$$




$\tilde{\varphi} := \psi^{-1} \circ \eta \circ \psi \in C^\infty$,
- isometry inside $S_{2\varepsilon}$,
- id on $S \setminus S_\varepsilon$
- But not area-pres. on $S$.

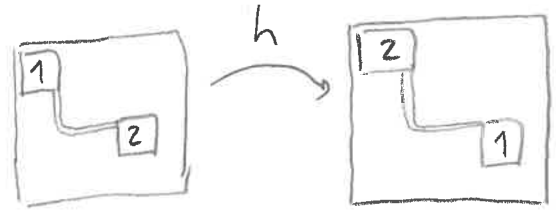
We have two symplectic forms $\Omega_0 = dx \wedge dy$ and $\tilde{\varphi}^* \Omega_0 = \Omega_1$

**Moser's trick**, [Moser], [McDuff-Salamon Exercise 3.2.6], [Berger 23], [FS 05]

Consider a smooth family of sympl. forms $\Omega_t = \Omega_0 + t(\Omega_1 - \Omega_0)$ s.t. $\Omega_t$ agree in a nbd of $\partial S$.
Then $\exists$ a smooth family of diffeos $\vartheta_t : S \to S$ s.t.
$\vartheta_0 = id$, $\vartheta_t|_{\partial S} = id$, $\vartheta_t^* \Omega_t = \Omega_0$
(In part., $\vartheta_1^* \Omega_1 = \Omega_0$).

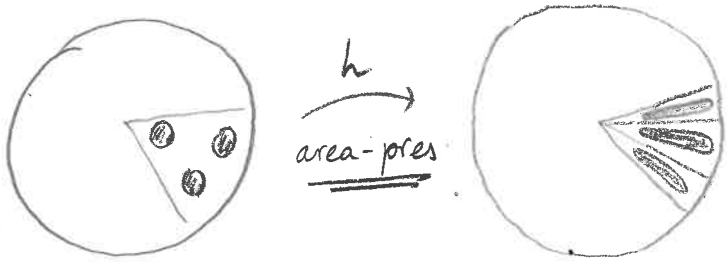
In our case: $\varphi := \vartheta_1 \tilde{\varphi}$ preserves the area.
Moreover, $\varphi = \tilde{\varphi}$ on the "TV-set" + "TV-box"

**Note** With a little more work:
$\exists\, h \in \mathcal{D}^\infty(A, \mu)$



**Th** [AK 70] Given any two collections of disjoint closed "$C^\infty$ sets", $(C_j)_{j=1,..n}$ and $(D_j)_{j=1,..n}$, on $M$

$\exists$ area-preserving $C^\infty$-smooth diffeo $h : C_j \to D_j \ \forall j = 1,..n$, and $h|_{\partial M} = id$.




**Rem** We used cut-off functions $\Rightarrow$ $C^\infty$ is a natural smoothness.

On $\mathbb{T}^d$ one can permute the partitions by area-pres. $C^\omega$-diffeos (explicit formulas) [Banerjee-Kunde] "Block-slide type of maps".