Lec. 1   **AbC method : main idea**

(I) Motivation (In which sense "anti-KAM"?)

a) Recall: • $\alpha \in DC(\gamma, \tau)$ $(\gamma > 0, \tau > 1)$ if $\forall p, q \in \mathbb{Z}$
$$|q\alpha - p| \geq \frac{\gamma}{q^\tau}.$$
• $\alpha \in$ Liouv. if $\alpha \in \mathbb{Q}^c \setminus \bigcup_{\gamma,\tau} DC(\gamma,\tau)$.

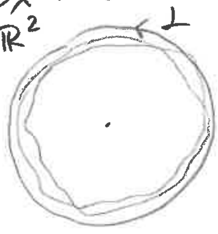
$\bigcup_\gamma DC(\gamma,\tau)$ has meas. 1; Liouv. is generic (contains a dense G_δ-set).

Usually, KAM (stability) $\rightsquigarrow$ DC twist
instability $\rightsquigarrow$ Liouv.   (Results depend on smoothness)

b) illustration

Ex: (Dichotomy)

$\mathbb{D} \subset \mathbb{R}^2$



Th (Herman [FKri])
$f \in \mathrm{Diff}^\infty(\mathbb{D}, \mu)$, μ Lcl
$f|_{\partial\mathbb{D}^2} = R_\alpha$
$\alpha \in DC$
Then $\exists$ positive meas. set of smooth inv. curves, accum. $\partial\mathbb{D}^2$ with dynam. $\underset{conj}{\sim} R_\alpha$
($\Rightarrow$ not ergodic)

Th. (FS)

$\forall \alpha \in$ Liouv., $\forall \varepsilon$
$\exists f \in \mathrm{Diff}^\infty(\mathbb{D}, \mu)$,
$f|_{\partial\mathbb{D}^2} = R_\alpha$, $f \overset{\varepsilon}{\sim} R_\alpha$
f is weakly mixing
($\Rightarrow$ ergodic)

$\boxed{\text{Constructed with AbC.}}$

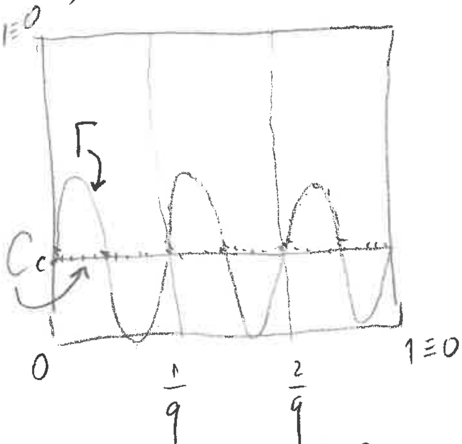
⌜We will see this type of results.⌟

c) History: Anosov, Katok '70 [AK]

We define abstract topological properties: ergodicity, minimality, ... Can they be realized by a C^∞ (or C^ω) Lebesgue measure preserving diffeo? In particular,

• ¿$\exists$? ergodic $f \in \mathrm{Diff}^\infty(\mathbb{D}, \mu)$ — yes, [AK]
• $\exists$? minimal but not ergodic f — yes [FH]
• $\exists$? mixing $f \in C^\omega(\mathbb{D}^2, \mu)$ — no [A
• $\exists$? mixing $f \in C^\infty(\mathbb{D}^2, \mu)$?

II a) Intuitive idea (on $\mathbb{T}^2$)



let $\alpha_0 = \frac{p}{q}$, $R_{\alpha_0}(x,y) = (x+\alpha_0, y) \mod 1$

$C = \{y = c\}$ — invar. set for R_α

Let h be $\frac{1}{q}$-periodic in x

(e.g. $h(x,y) = (x, y + a \sin 2\pi q x)$)

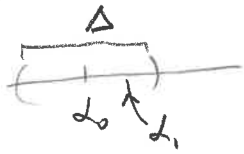
Then $f_0 := h^{-1} \circ R_{\alpha_0} \circ h = R_{\alpha_0}$

Let $\alpha_1 = \frac{p_1}{q_1} \approx \frac{p_0}{q_0}$, let $\underline{f_1} = h^{-1} \circ R_{\alpha_1} \circ h \approx \underline{f_0} = R_{\alpha_0}$

b) Lemma (*)

$\forall \varepsilon > 0$, $\forall \alpha_0 = \frac{p}{q}$, $\forall h$ $\frac{1}{q}$-periodic

$\forall k$



$\exists \Delta = \Delta(\|h\|_k, \varepsilon, k, m)$ s.t. $\forall \alpha_1 \in \Delta$

• $\|h^{-1} \circ R_{\alpha_1} \circ h - R_{\alpha_0}\|_k < \varepsilon$

•• $\|\underbrace{h^{-1} \circ R^j_{\alpha_1} \circ h}_{f_1^j} - R^j_{\alpha_0}\|_0 < \varepsilon$ $\forall j = 0, \dots m$ (in part., can take $m = q$)

c) Dynamics:

Let $\Gamma = h^{-1}(C)$. Then $\underline{f_1}(\Gamma) = h^{-1} \circ R_{\alpha_1} \circ h\,(h^{-1}(C)) = \underline{\Gamma}$.

Moreover, if $\alpha_1 \notin \mathbb{Q}$, f_1 is dense on Γ.

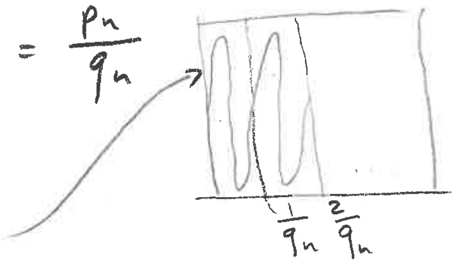
Γ = grapf of $\gamma(x) = c - a \sin 2\pi q x$.

By an $\underline{\varepsilon\text{-small}}$ perturb. of $R_{\alpha_0 = \frac{p}{q}}$, we got a prescribed ($\frac{1}{q}$-periodic) <u>very different</u> dynamics. Moreover, $f_1^j(z) \overset{\varepsilon}{\approx} f_0^j(z)$ for a prescribed long # of iterates.

d) Iterate the construction. (for example, in C^∞)

Take $k_n = 2^n$, $\varepsilon_n = \frac{1}{2^n}$, $m_n = q_n$

Given $f_{n-1} = H_{n-1}^{-1} \circ R_{\alpha_n} \circ H_{n-1}$, $\alpha_n = \frac{p_n}{q_n}$



Construct h_n $\frac{1}{q_n}$-periodic, area-pres.,
with "crazy dynamics"

Let $f_n = H_{n-1}^{-1} \circ h_n^{-1} \circ R_{\alpha_{n+1}} \circ h_n \circ H_{n-1}$

By Lem. ⊛, $\exists\, \alpha_{n+1} \approx \alpha_n$ such that:

- $\| f_n - f_{n-1} \|_{k_n} < \varepsilon_n$
- $\| f_n^j - f_{n-1}^j \|_0 < \varepsilon_n \quad \forall j = 0, \dots q_n$ (Note: $f_{n-1}^{q_n} = id$
- $f_n^j(z)$ for $j = 1, \dots q_{n+1}$ fills out Γ_n "very densely".

e) Convergence (to $f \in \mathrm{Diff}^\infty(\mathbb{T}^2, \mu)$) occurs since
$(f_n)_n$ form a Cauchy sequence (in $\|\cdot\|_{k_n}$ for any n)

Analogousely, can prove convergence in $\mathrm{Diff}_\varrho^\omega$.

f) Dynamics of f. For any n we have:

$$\| f^j - f_{n-1}^j \|_0 \le \underbrace{\| f_{n-1}^j - f_n^j \|_0}_{\substack{< \varepsilon_n \\ \forall j = 0, \dots q_n}} + \underbrace{\| f_n^j - f_{n+1}^j \|_0}_{\substack{< \varepsilon_{n+1} \\ \forall j = 0, \dots q_{n+1} \\ \gg q_n}} + \dots < \underbrace{\sum_{\ell=n}^\infty \varepsilon_\ell}_{\text{small}}$$

So, $f^j(x)$ follows $f_{n-1}^j(x)$ for $j = 1, \dots q_n$;
Recall that $f_{n-1}^{q_n} = id$

III Some applications

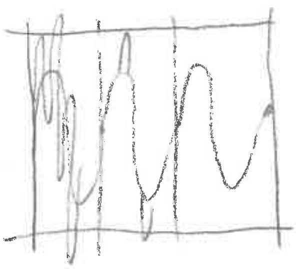
Th (Partly in [KH], [Katok-Robinson]; here I follow [S])

Let $\varepsilon > 0, \delta > 0$, $h_n = (x, y + c_n \sin 2\pi q_n x)$, $\alpha_n = \frac{p_n}{q_n}$, (q_n) grow sufficiently fast (depending on ε, (c_n)), f_n constructed as above.

Then $\exists\, f = \lim\limits_{n\to\infty} f_n \in \mathrm{Diff}_\delta^{\,\omega}(\mathbb{T}^2, \mu)$, $\|f - R_\alpha\|_\delta < \varepsilon$ s.t.

1) If $\sum^{\infty} c_j = \infty$, f is minimal
2) If $c_n = \frac{1}{n}$ $\forall n$, f is minimal, not ergodic
3) If c_n "grow sufficiently fast", f is ergodic.

Why 1) In our case $H_{n-1}(x, y) = (x, y + \sum^{n-1} c_j \sin 2\pi q_j x)$



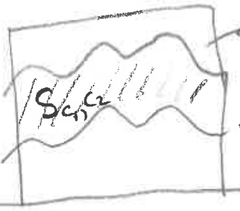
Γ_{n-1} = graph of $\gamma_{n-1}(x) = c_0 + \sum_{j=1}^{n-1} c_j \cos 2\pi q_j x$
= invariant curve for $f_{n-1} = H_{n-1}^{-1} \circ R_{\frac{p_n}{q_n}} \circ H_{n-1}$.

If q_n suff. large, • $(f_{n-1}^j(x))$, $j = 1, \ldots q_n$, is ε_n-dense $\forall x$

• $\|f^j - f_{n-1}^j\|_0 < \varepsilon_n$ for all $j = 1, \ldots q_n$

$\Rightarrow (f^j(x))$, $j = 1, \ldots q_n$, is $2\varepsilon_n$-dense $\forall x$

Why 2) $\left(\frac{1}{n}\right) \in l_2 \Rightarrow \sum^{\infty} \frac{1}{n} \cos 2\pi q_n x$ is a Fourier-series of an L^2-function. By Carleson's theorem, it converges (to γ) a.e., so $\gamma \overset{a.e.}{=} \sum^{\infty} \frac{1}{n} \cos 2\pi q_n x$ is measurable.



S_{c_1,c_2} is an invariant set

$\mu(S_{c_1,c_2}) \notin \{0, 1\}$.

$\Rightarrow f$ is not ergodic.